Problemy Matematyczne
15 (1997), 57–62

The Minimal Polynomials of $\cos(2\pi/n)$ over Q

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1. Introduction

It is well known that ζ, a primitive n-th root of unity, satisfies the equation

$$x^n - 1 = 0. \tag{1}$$

In this work we study the minimal polynomials of the real part of ζ, i.e. of $\cos(2\pi/n)$, over the rationals. This number also plays an important role in some geometrical calculations with 3-dimensional solid figures and in the theory of regular star polygons. We use a paper of Watkin and Zeitlin ([2]) to produce further results. In the calculations, we use another class of polynomials called Chebysheff polynomials. They are recalled here and form the subject of section 2. By means of them we obtain several cerurrence formulae for the minimal polynomials of $\cos(2\pi/n)$.

2. Chebysheff polynomials

Definition 1 *Let $n \in \mathsf{N} \cup \{0\}$. Then the n-th Chebysheff polynomial, denoted by $T_n(x)$, is defined by*

$$T_n(x) = \cos(n \cdot \arccos x), \quad x \in \mathsf{R}, \quad |x| \leq 1, \tag{2}$$

or

$$T_n(\cos\theta) = \cos n\theta, \quad \theta \in \mathsf{R}, \quad (\theta = \arccos \mathrm{x} + 2\mathrm{k}\pi, \ \ \mathrm{k} \in \mathsf{Z}). \qquad (3)$$

We will drop the conditions on x and θ since they always apply.

Example 1

Let us compute $T_7(x)$. Let $\theta = \arccos x$. Then

$$T_7(x) = T_7(\cos\theta) = \cos 7\theta = Re\left((\cos\theta + i\sin\theta)^7\right),$$

hence

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x. \qquad (4)$$

The first few Chebysheff polynomials are given below:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\ T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \end{aligned} \qquad (5)$$

There is the following recurrence formulae for T_n :

Lemma 1 *Let* $n \in \mathsf{N}$. *Then*

$$T_{n+1}(x) = 2T_n(x) - T_{n-1}(x). \qquad (6)$$

This is a result of some trigonometric identities.

It follows that the degree of $T_n(x)$ is n and its leading coefficient is 2^{n-1}. This can be proven by induction on n. There are some identities involving Chebysheff polynomials:

Lemma 2

$$(x-1)\cdot(T_{2n+1}(x)-1)=(T_{n+1}(x)-T_n(x))^2, \quad (7)$$

$$2\cdot\left(x^2-1\right)\cdot(T_{2n}(x)-1)=(T_{n+1}(x)-T_{n-1}(x))^2 \quad if \quad n\neq 0. \quad (8)$$

3. The minimal polynomials of $\cos(2\pi/n)$ over Q

For $n \in \mathrm{N}$, we denote the minimal polynomial of $\cos(2\pi/n)$ over Q by $\Psi_n(x)$. Then Watkins and Zeitlin ([2]) used the Galois theory to show the following:

Lemma 3 $\deg\Psi_0(x)=\deg\Psi_2(x)$ *and* $\deg\Psi_n(x)=\varphi(n)/2$ *for* $n\geq 3$, *where* φ *denotes the Euler* φ*-function.*

Since $\sigma(\xi)=\xi^k$ is an automorphism of $\mathrm{Q}(\xi)$ over Q then the roots of $\Psi_m(x)$ over Q are

$$\{\cos(2k\pi/m) \ : \ k\in \mathrm{N} \quad (k,m)=1, \quad k\leq n, where\ n=[m/2]\}.$$

In [2] the relations between Ψ_n's and T_n's are given as follows:

Lemma 4 *Let* $m \in \mathrm{N}$ *and let* n *be as above. Then*

$$T_{n+1}(x)-T_n(x)=2^n\cdot\prod_{d|m}\Psi_d(x) \quad (9)$$

if m *is odd and*

$$T_{n+1}(x)-T_{n-1}(x)=2^n\cdot\prod_{d|m}\Psi_d(x) \quad (10)$$

if m *is even.*

By means of Lemma 4 we obtain the following formulae for the minimal polynomial $\Psi_n(x)$.

Theorem 1 *Let $m \in \mathbb{N}$ and let n be as above. Then*

(a) $\Psi_1(x) = x - 1$, $\Psi_2(x) = x + 1$,

(b) *If m is odd and prime then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^n(x-1)}.$$

(c) *If $4|m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n/2} \cdot \left(T_{\frac{n}{2}+1}(x) - T_{\frac{n}{2}-1}(x)\right) \cdot \prod_{d|m, d \neq m, d \nmid \frac{m}{2}} \Psi_d(x)}.$$

(d) *If m is even and $m/2$ is odd, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n-n'} \cdot (T_{n'+1}(x) - T_{n'}(x)) \cdot \prod_{d|m, d \neq m, 2|d} \Psi_d(x)},$$

where

$$n' = \frac{m/2 - 1}{2}.$$

(e) *Let m be odd and let p be a prime dividing m. If $p^2|m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'} \cdot (T_{n'+1}(x) - T_{n'}(x))},$$

where

$$n' = \frac{m/p - 1}{2}.$$

If $p^2|m$, then

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'} \cdot (T_{n'+1}(x) - T_{n'}(x)) \, \Psi_p(x)},$$

where

$$n' = \frac{m/p - 1}{2}.$$

Proof. (a) If $m = 1$ then $n = 0$. Therefore by Lemma 4 (9),

$$\Psi_1(x) = \frac{1}{2^0}\left(T_1(x) - T_0(x)\right) = x - 1.$$

Similarly for $m = 2$, $\Psi_2(x) = x + 1$.

(b) Let m be an odd prime. Then $n = \frac{m-1}{2}$ and by Lemma 4 (9), $T_{n+1}(x) - T_n(x) = 2^n \cdot \Psi_1(x) \cdot \Psi_m(x)$ and the result follows.

The proofs of (c), (d) and (e) are similar. Hence we shall prove (d) only. Let m be even and $m/2$ be odd. By Lemma 4

$$T_{n+1}(x) - T_n(x) = 2^n \cdot \prod_{d|m} \Psi_d(x)$$

since m is even and

$$T_{n'+1}(x) - T_{n'}(x) = 2^n \cdot \prod_{d|m} \Psi_d(x)$$

since $m/2$ is odd, where

$$n' = \frac{m/2 - 1}{2}.$$

Now

$$\frac{\prod_{d|m} \Psi_d(x)}{\prod_{d|m/2} \Psi_d(x)} = \prod_{d|m,\ 2|d} \Psi_d(x) = \Psi_m(x) \cdot \prod_{d|m,\ d\neq m,\ 2|d} \Psi_d(x)$$

and hence the result follows.

Example 2 (a)

Let us determine the minimal polynomial of $\cos(2\pi/5)$ over Q. By theorem 1 (b)

$$\Psi_5(x) = \frac{T_3(x) - T_2(x)}{2^2(x-1)} = x^2 + \frac{1}{2}x - \frac{1}{4}.$$

(b) Determine the minimal polynomial of $\cos(2\pi/10)$ over Q. By theorem 1 (d), $n' = 2$ and

$$\Psi_{10}(x) = \frac{T_6(x) - T_4(x)}{2^3} \cdot (T_3(x) - T_2(x)) \cdot \prod_{d|10, d\neq 10, 2|d} \Psi_d(x) =$$

$$x^2 - \frac{1}{2}x - \frac{1}{4}.$$

Therefore we have established some recurrence formulae for the minimal polynomial $\Psi_m(x)$ of $\cos(2\pi/m)$ over Q.

REFERENCES

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